Base-flow, Global modes
Adjoint operator,
Adjoint global modes

Denis Sipp

ONERA / DAAA
Meudon
denis.sipp@onera.fr

Outline

- Governing equations
- Asymptotic development
  - Order ϵ^0 : Base-flow
  - Order ϵ^1 : Global modes
- Bi-orthogonal basis and adjoint global modes
- Adjoint operator
  - Definition
  - Adjoint global modes as solutions of adjoint eigen-problem
- Adjoint linearized Navier-Stokes operator
  - Adjoint of linearized advection operator
  - Adjoint of Stokes operator
  - Adjoint global modes of cylinder flow

Governing equations (Ginzburg-Landau, GL)

Forced nonparallel, nonlinear Ginzburg-Landau equation:

$$\partial_t w + U\partial_x w + w|w|^2 = \mu(x)w + \gamma\partial_{xx}w + f(x,t)$$
$$\mu(x) := i\omega_0 + \mu_0 - \underbrace{\gamma\chi^4}_{=\mu_2} x^2$$
$$|w| \to 0 \text{ as } x \to \pm\infty$$

and $U, \gamma, \ \omega_0, \mu_0, \mu_2$ are positive real constant, $f(x,t)$ a "weak" forcing

Can be recast into:

$$\partial_t w + \mathcal{L}w + \mathcal{N}(w) = f(x,t)$$

where

$$\mathcal{L} := U\partial_x - \mu(x) - \gamma\partial_{xx}$$
$$\mathcal{N}(w) := w|w|^2$$

Governing equations (Ginzburg-Landau, GL)

Advection:

$$\partial_t w + U\partial_x w = 0$$

Diffusion:

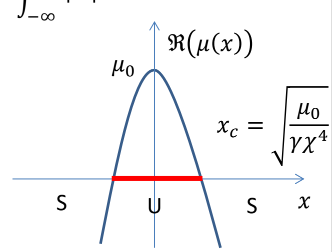
$$\partial_t w = \gamma\partial_{xx}w$$

Dissipative non-linearity:

$$\partial_t w + w|w|^2 = 0$$
$$\partial_t\left(\frac{|w|^2}{2}\right) + |w|^4 = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{dt}}\left(\int_{-\infty}^{+\infty}\frac{|w|^2}{2}dx\right) = -\int_{-\infty}^{+\infty}|w|^4 dx < 0$$

Localized in-space instability term:

$$\partial_t w = \mu(x)w$$
$$\mu(x) := i\omega_0 + \mu_0 - \gamma\chi^4 x^2$$



Forcing:

$$\partial_t w = f(x,t)$$

Governing equations (Navier-Stokes, NS)

Incompressible Navier-Stokes equations:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = -\partial_x p + \nu\left(\partial_{xx} u + \partial_{yy} u\right) + f \\ \partial_t v + u\partial_x v + v\partial_y v = -\partial_y p + \nu\left(\partial_{xx} v + \partial_{yy} v\right) + g \\ -\partial_x u - \partial_y v = 0 \end{cases}$$

Can be recast into:

$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = f$$

where:

$$w = \begin{pmatrix} u \\ p \end{pmatrix} \quad f = \begin{pmatrix} f \\ 0 \end{pmatrix}$$

$$\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\mathcal{N}(w_1, w_2) = \begin{pmatrix} u_1 \cdot \nabla u_2 + u_2 \cdot \nabla u_1 \\ 0 \end{pmatrix}$$

$$\mathcal{L} = \begin{pmatrix} -\nu\Delta() & \nabla() \\ -\nabla \cdot () & 0 \end{pmatrix}$$

Boundary conditions: Dirichlet, Neumann, Mixed

Governing equations (Navier-Stokes, NS)

a) $\mathcal{N}(w_1, w_2) = \mathcal{N}(w_2, w_1)$

b) $\frac{1}{2}\mathcal{N}(w_0 + \epsilon\delta w, w_0 + \epsilon\delta w) = \frac{1}{2}\mathcal{N}(w_0, w_0) + \epsilon \underbrace{\mathcal{N}(w_0, \delta w)}_{\substack{\text{Jacobian} \\ = \mathcal{N}_{w_0}\delta w}} + \frac{\epsilon^2}{2}\underbrace{\mathcal{N}(\delta w, \delta w)}_{\text{Hessian}} + \cdots$

c) $\mathcal{N}_{w_0}\delta w = \mathcal{N}(w_0, \delta w) = \begin{pmatrix} \delta u \cdot \nabla u_0 + u_0 \cdot \nabla \delta u \\ 0 \end{pmatrix}$

d) $\mathcal{B}w = \mathcal{B}\begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} u \\ 0 \end{pmatrix}$

e) $\partial_t u + u \cdot \nabla u = -\nabla p + \nu\nabla^2 u \Rightarrow -\nabla^2 p = \nabla \cdot (u \cdot \nabla u), \partial_n p = \nu\nabla^2 u \cdot n$ on solid walls. Hence, p is a function of u and should not be considered as a degree of freedom of the flow.

f) Scalar-product: $< w_1, w_2 > = \iint (u_1^* u_2 + v_1^* v_2) dxdy = \iint (w_1 \cdot \mathcal{B}w_2) dxdy$ so that $\sqrt{\{< w, w >\}}$ is the energy.

Outline

- Governing equations
- Asymptotic development
  - Order ϵ^0 : Base-flow
  - Order ϵ^1 : Global modes
- Bi-orthogonal basis and adjoint global modes
- Adjoint operator
  - Definition
  - Adjoint global modes as solutions of adjoint eigen-problem
- Adjoint linearized Navier-Stokes operator
  - Adjoint of linearized advection operator
  - Adjoint of Stokes operator
  - Adjoint global modes of cylinder flow

Asymptotic development (GL)

Solution:

$$w(t) = w_0 + \epsilon w_1(t) + \cdots \quad \text{with } \epsilon \ll 1$$

Governing equations (with $f(x,t) = 0$):

$$\partial_t w + \mathcal{L}w + \mathcal{N}(w) = 0$$

Introduce solution into governing eq:

$$\partial_t(w_0 + \epsilon w_1(t) + \cdots) + \mathcal{L}(w_0 + \epsilon w_1(t) + \cdots) + \mathcal{N}(w_0 + \epsilon w_1(t) + \cdots) = 0$$

$$\Rightarrow \begin{cases} \mathcal{L}w_0 + \mathcal{N}(w_0) = 0 \text{ and } |w_0| \to 0 \text{ as x} \to \pm\infty \text{ at order } O(1) \Rightarrow w_0 = 0 \\ \partial_t w_1 + \mathcal{L}w_1 = 0 \text{ and } |w_1| \to 0 \text{ as x} \to \pm\infty \text{ at order } O(\epsilon) \end{cases}$$

$$\mathcal{L} := U\partial_x - \mu(x) - \gamma\partial_{xx}$$
$$\mathcal{N}(w) := w|w|^2$$
$$|w| \to 0 \text{ as x} \to \pm\infty$$

Asymptotic development (NS)

Solution:

$$w(t) = w_0 + \epsilon w_1(t) + \cdots \quad \text{with } \epsilon \ll 1$$

Governing equations:

$$\mathcal{B}\partial_t w + \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w = 0$$

Introduce solution into governing eq::

$$\mathcal{B}\partial_t(w_0+\epsilon w_1 + \cdots) + \frac{1}{2}\mathcal{N}(w_0 + \epsilon w_1 + \cdots, w_0 + \epsilon w_1 + \cdots) + \mathcal{L}(w_0+\epsilon w_1 + \cdots) = 0$$

$$\Rightarrow \begin{cases} \frac{1}{2}\mathcal{N}(w_0,w_0) + \mathcal{L}w_0 = 0 \text{ at order } O(1) \\ \mathcal{B}\partial_t w_1 + \frac{1}{2}\underbrace{[\mathcal{N}(w_1,w_0) + \mathcal{N}(w_0,w_1)]}_{\mathcal{N}_{w_0}w_1} + \mathcal{L}w_1 = 0 \text{ at order } O(\epsilon) \end{cases}$$

Oder ϵ^0: Base-flow

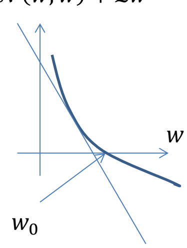
Definition:

$$w(t) = w_0 + \epsilon w_1(t) + \cdots$$

Non-linear equilibrium point :

$$\frac{1}{2}\mathcal{N}(w_0,w_0) + \mathcal{L}w_0 = 0$$

$$F(w) = \frac{1}{2}\mathcal{N}(w,w) + \mathcal{L}w$$



How to compute a base-flow ?

Newton iteration: guess $\Rightarrow w_0$

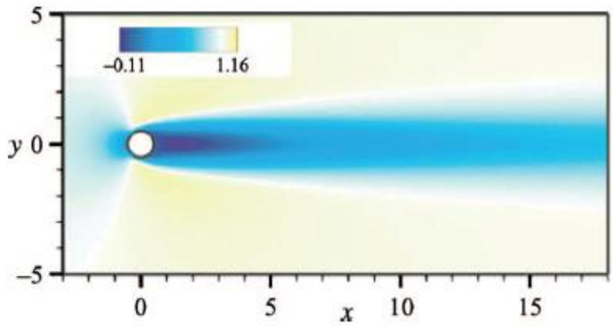
$$\frac{1}{2}\mathcal{N}(w_0 + \delta w_0, w_0 + \delta w_0) + \mathcal{L}(w_0+\delta w_0) = 0$$

Linearization:

$$\mathcal{N}(w_0,\delta w_0) + \mathcal{L}\delta w_0 = -\frac{1}{2}\mathcal{N}(w_0,w_0) - \mathcal{L}w_0$$

$$\Rightarrow \delta w_0 = \left(\mathcal{N}_{w_0} + \mathcal{L}\right)^{-1}\left(-\frac{1}{2}\mathcal{N}(w_0,w_0) - \mathcal{L}w_0\right)$$

Oder ϵ^0: Base-flow
The case of cylinder flow



$Re = 47$

Streamwise velocity field of base-flow.

Order ϵ^1: Global modes (GL)

$w(t) = w_0 + \epsilon w_1(t) + \cdots$

Linear governing equation:

$$\partial_t w_1 + \mathcal{L} w_1 = 0 \text{ and } |w_1| \to 0 \text{ as } x \to \pm\infty$$

Solution w_1 under the form:

$$w_1 = e^{\lambda t} \hat{w}$$

This leads to :

$$\lambda \hat{w} + \mathcal{L} \hat{w} = 0$$

Eigenvalue:

$$\lambda = \sigma + i\omega$$

Eigenvector:

$$\hat{w} = \hat{w}_r + i \hat{w}_i$$

Order ϵ^1: Global modes (GL)

Show that $\hat{w}(x) = \zeta e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$ with $\zeta = \sqrt{\chi}\pi^{-\frac{1}{4}} e^{-\frac{U^2}{2\gamma^2\chi^2}}$ verifies $\lambda\hat{w} + \mathcal{L}\hat{w} = 0$. What is the eigenvalue λ associated to this eigenvector? The constant ζ has been selected so that $\langle \hat{w}, \hat{w} \rangle = 1$, where:

$$\langle w_a, w_b \rangle = \int_{-\infty}^{+\infty} w_a(x)^* w_b(x) dx$$

Show that the eigenmode is marginnaly stable if $\mu_0 = \mu_c = \frac{U^2}{4\gamma} + \gamma\chi^2$. What is the frequency of this mode?

Nota: $\left(\lambda_n = \mathrm{i}\omega_0 + \mu_0 - \frac{U^2}{4\gamma} - (2n+1)\gamma\chi^2, \hat{w}_n = \zeta_n H_n(\chi x) e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}\right)$ are all the eigenvalues/eigenvectors of $\mathcal{L}$, H_n being Hermite polynomials.

Order ϵ^1: Global modes (GL)

$$\partial_x \hat{w} = \zeta \left(\frac{U}{2\gamma} - \chi^2 x\right) e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$$

$$\partial_{xx} \hat{w} = \zeta \underbrace{\left(-\chi^2 + \left(\frac{U}{2\gamma} - \chi^2 x\right)\left(\frac{U}{2\gamma} - \chi^2 x\right)\right)}_{= -\chi^2 + \frac{U^2}{4\gamma^2} + \chi^4 x^2 - \frac{U}{\gamma}\chi^2 x} e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$$

$$\mathcal{L}\hat{w} = U\partial_x\hat{w} - i\omega_0\hat{w} - \mu_0\hat{w} + \gamma\chi^4 x^2\hat{w} - \gamma\partial_{xx}\hat{w}$$

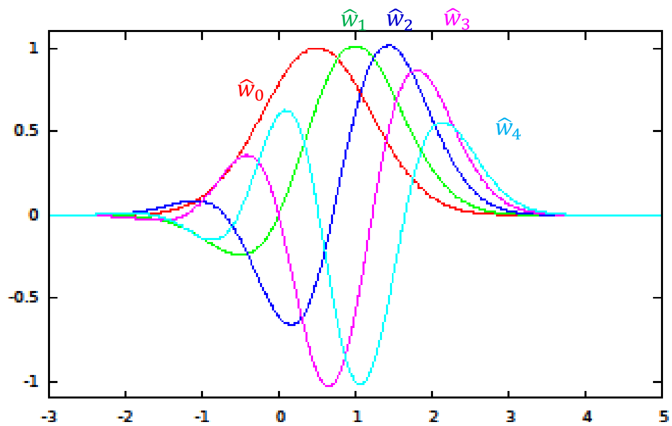
$$= \left[U\left(\frac{U}{2\gamma} - \chi^2 x\right) - i\omega_0 - \mu_0 + \gamma\chi^4 x^2 - \gamma\left(-\chi^2 + \frac{U^2}{4\gamma^2} + \chi^4 x^2 - \frac{U}{\gamma}\chi^2 x\right)\right]\zeta e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}} = -\underbrace{\left(i\omega_0 + \mu_0 - \frac{U^2}{4\gamma} - \gamma\chi^2\right)}_{\lambda}\hat{w}$$

Hence:

$$\lambda = i\omega_0 + \mu_0 - \frac{U^2}{4\gamma} - \sqrt{\frac{\gamma\mu_2}{2}}$$

At criticality: $\mu_c - \frac{U^2}{4\gamma} - \sqrt{\frac{\gamma\mu_2}{2}} = 0 \Rightarrow \lambda_c = i\omega_0$

Order ϵ^1: Global modes (GL)



Order ϵ^1: Global modes (NS)

$w(t) = w_0 + \epsilon w_1(t) + \cdots$

Linear governing equation:

$$\mathcal{B}\partial_t w_1 + \mathcal{N}_{w_0} w_1 + \mathcal{L} w_1 = 0$$

Solution w_1 under the form:

$$w_1 = e^{\lambda t}\widehat{w} + \text{c.c}$$

This leads to :

$$\lambda \mathcal{B}\widehat{w} + \left(\mathcal{N}_{w_0} + \mathcal{L}\right)\widehat{w} = 0$$

Eigenvalue:

$$\lambda = \sigma + i\omega$$

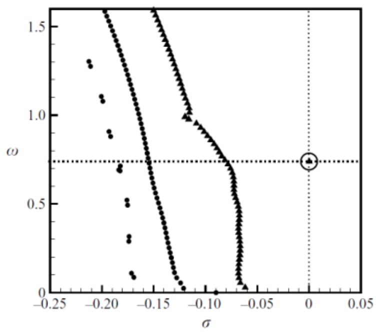
Eigenvector:

$$\widehat{w} = \widehat{w}_{\mathrm{r}} + \mathrm{i}\widehat{w}_i$$

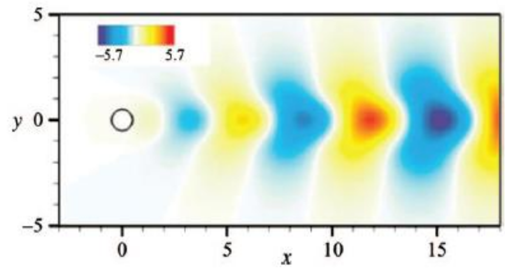
Real solution:

$$w_1 = e^{\lambda t}\widehat{w} + \text{c.c} = 2e^{\sigma t}(\cos\omega t\, \widehat{w}_r - \sin\omega t\, \widehat{w}_i)$$

Order ϵ^1: Global modes
Case of cylinder flow



Spectrum $Re = 47$



Real part of cross-stream velocity field
Marginal eigenmode

Order ϵ^1: Global modes
How to compute global modes ?

Eigenvalue problem solved with shift-invert strategy:
- Power method, easy to find largest magnitude eigenvalues of $Ax = \lambda x$. For this, evaluate $A^n x_0$
- To find eigenvalues of A closest to zero, search largest magnitude eigenvalues of A^{-1}: $A^{-1}x = \lambda^{-1}x$. For this, evaluate$(A^{-1})^n x_0$
- To find eigenvalues of A closest to s, search largest magnitude eigenvalues of $(A - sI)^{-1}$: $(A - sI)^{-1}x = (\lambda - s)^{-1}x$. For this, evaluate$\left((A - sI)^{-1}\right)^n x_0$
- Instead of power-method, use Krylov subspaces -> Arnoldi technique
- Cost of algorithm = cost of several complex matrix inversions

Outline

- Governing equations
- Asymptotic development
  - Order ϵ^0 : Base-flow
  - Order ϵ^1 : Global modes
- Bi-orthogonal basis and adjoint global modes
- Adjoint operator
  - Definition
  - Adjoint global modes as solutions of adjoint eigen-problem
- Adjoint linearized Navier-Stokes operator
  - Adjoint of linearized advection operator
  - Adjoint of Stokes operator
  - Adjoint global modes of cylinder flow

Bi-orthogonal basis and adjoint global modes

In finite dimension

Global modes:

$$A\hat{w}_i = \lambda_i \hat{w}_i$$

The eigenvectors $\hat{w}_i$ form a basis:

$$w = \sum_i \alpha_i \hat{\mathrm{w}}_\mathrm{i}$$

Definition of adjoint global modes: with <> as a given scalar-product (say $< w_1, w_2 > = w_1^* w_2$), there exists for each α_i a unique $\tilde{w}_i$ such that $\alpha_i =< \tilde{w}_i, w >$ for all w. The adjoint global modes are the structures $\tilde{w}_i$. In the following: $\langle \hat{w}_i, \hat{w}_i \rangle = 1$.

Properties:

- $\tilde{w}_k$ and $\hat{\mathrm{w}}_\mathrm{j}$ are bi-orthogonal bases: they verify $\hat{w}_j = \sum_i < \tilde{w}_i, \hat{w}_j > \hat{\mathrm{w}}_\mathrm{i}$ and so $< \tilde{w}_k, \hat{\mathrm{w}}_\mathrm{j} > = \delta_{kj}$
- Cauchy-Schwarz: $1 = |< \tilde{w}_i, \hat{\mathrm{w}}_\mathrm{i} >| \leq < \tilde{w}_i, \tilde{w}_i >^{\frac{1}{2}} < \hat{w}_i, \hat{w}_i >^{\frac{1}{2}}$

Hence: $< \tilde{w}_i, \tilde{w}_i >^{\frac{1}{2}} \geq 1$ and $\cos \mathrm{angle}(\tilde{w}_i, \hat{w}_i) = \frac{1}{< \tilde{w}_i, \tilde{w}_i >^{\frac{1}{2}}}$

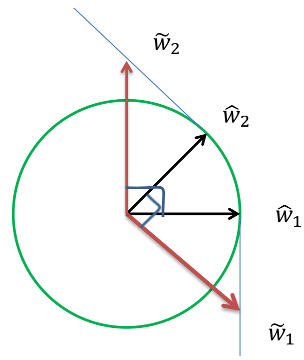
Bi-orthogonal basis and adjoint global modes

In finite dimension

$w = (\tilde{w}_1 \cdot w)\hat{w}_1 + (\tilde{w}_2 \cdot w)\hat{w}_2$

Def of $\tilde{w}_1$:
$\tilde{w}_1 \cdot \hat{w}_1 = 1$
$\tilde{w}_1 \cdot \hat{w}_2 = 0$

Def of $\tilde{w}_2$:
$\tilde{w}_2 \cdot \hat{w}_2 = 1$
$\tilde{w}_2 \cdot \hat{w}_1 = 0$



Bi-orthogonal basis and adjoint global modes (GL)

Global modes:

$$\lambda_i \hat{w}_i + \mathcal{L}\hat{w}_i = 0$$

The eigenvectors $\hat{w}_i$ form a basis:

$$w = \sum_i \alpha_i \hat{w}_i$$

Definition of adjoint global modes: with <> as a given scalar-product, there exists for each α_i a unique $\tilde{w}_i$ such that $\alpha_i = < \tilde{w}_i, w >$ for all w. The adjoint global modes are the structures $\tilde{w}_i$. In the following: $< \hat{w}_i, \hat{w}_i > = 1$.

Properties:

- $\tilde{w}_k$ and $\hat{w}_j$ are bi-orthogonal bases: they verify $\hat{w}_j = \sum_i < \tilde{w}_i, \hat{w}_j > \hat{w}_i$ and so $< \tilde{w}_k, \hat{w}_j > = \delta_{kj}$
- Cauchy-Schwarz : $1 = |< \tilde{w}_i, \hat{w}_i >| \leq < \tilde{w}_i, \tilde{w}_i >^{\frac{1}{2}} < \hat{w}_i, \hat{w}_i >^{\frac{1}{2}}$. Hence: $< \tilde{w}_i, \tilde{w}_i >^{\frac{1}{2}} \geq 1$ and $\cos \text{angle}(\tilde{w}_i, \hat{w}_i) = \frac{1}{< \tilde{w}_i, \tilde{w}_i >^{\frac{1}{2}}}$

Bi-orthogonal basis and adjoint global modes (NS)

Global modes:

$$\lambda_i \mathcal{B}\widehat{w}_i + \left(\mathcal{N}_{w_0} + \mathcal{L}\right)\widehat{w}_i = 0$$

The eigenvectors $\widehat{w}_i$ form a basis:

$$w = \sum_i \alpha_i \widehat{\mathrm{w}}_\mathrm{i}$$

Definition of adjoint global modes: with <> as a given scalar-product, there exists for each α_i a unique $\widetilde{w}_i$ such that $\alpha_i = < \widetilde{w}_i, \mathcal{B}w >$ for all w. The adjoint global modes are the structures $\widetilde{w}_i$. In the following: $< \widehat{w}_i, \mathcal{B}\widehat{w}_i > = 1$.

Properties:

- $\widetilde{w}_k$ and $\widehat{\mathrm{w}}_\mathrm{j}$ are bi-orthogonal bases: they verify $\widehat{w}_j = \sum_i < \widetilde{w}_i, \mathcal{B}\widehat{w}_j > \widehat{\mathrm{w}}_\mathrm{i}$ and so $< \widetilde{w}_k, \mathcal{B}\widehat{\mathrm{w}}_\mathrm{j} > = \delta_{kj}$
- Cauchy-Schwarz : $1 = |< \widetilde{w}_i, \mathcal{B}\widehat{\mathrm{w}}_\mathrm{i} >| \leq < \widetilde{w}_i, \mathcal{B}\widetilde{w}_i >^{\frac{1}{2}} < \widehat{w}_i, \mathcal{B}\widehat{w}_i >^{\frac{1}{2}}$. Hence: $< \widetilde{w}_i, \mathcal{B}\widetilde{w}_i >^{\frac{1}{2}} \geq 1$ and $\cos \text{angle}(\widetilde{w}_i, \widehat{w}_i) = \frac{1}{< \widetilde{w}_i, \mathcal{B}\widetilde{w}_i >^{\frac{1}{2}}}$

Outline

Adjoint operator
Definition

Definition of adjoint operator:
Let $\langle w_1, w_2 \rangle$ be a scalar product and $\mathcal{A}$ a linear operator.
The adjoint operator of $\mathcal{A}$ verifies $\langle w_1, \mathcal{A} w_2 \rangle = \langle \tilde{\mathcal{A}} w_1, w_2 \rangle$
whatever w_1 and w_2.

Adjoint operator
(In finite dimension)

Space:

$$w \in \mathbb{C}^N$$

Scalar-product:

$$< w_1, w_2 > = w_1^* Q w_2$$

with Q a Hermitian matrix $Q^* = Q$.

Linear operator: $\mathcal{A}$ matrix.

Adjoint operator:

$$< w_1, \mathcal{A} w_2 > = w_1^* Q \mathcal{A} w_2 = w_1^* Q \mathcal{A} Q^{-1} Q w_2 = (Q^{-1} \mathcal{A}^* Q w_1)^* Q w_2 =< \tilde{\mathcal{A}} w_1, w_2 >$$

with $\tilde{\mathcal{A}} = Q^{-1} \mathcal{A}^* Q$

If $Q = I$, then $\tilde{\mathcal{A}} = \mathcal{A}^*$

Adjoint operator (GL)

Show that the operator

$$\mathcal{L} = U\partial_x - \mu(x) - \gamma\partial_{xx}$$

exhibits the following adjoint

$$\tilde{\mathcal{L}} = -U\partial_x - \overline{\mu(x)} - \gamma\partial_{xx}$$

when considering the following scalar-product

$$\langle w_a, w_b \rangle = \int_{-\infty}^{+\infty} \overline{w_a(x)} w_b(x) dx$$

and the boundary conditions:

$$|w| \to 0 \text{ as x} \to \pm\infty$$

Adjoint operator (GL)

$$\mathcal{L} = U\partial_x - \mu(x) - \gamma\partial_{xx}$$

$$< u, \mathcal{L}v > = < \tilde{\mathcal{L}}u, v >$$

$$\int_{-\infty}^{\infty} \bar{u}(U\partial_x v - \mu(x)v - \gamma\partial_{xx}v)dx = \int_{-\infty}^{\infty} (\bar{u}U\partial_x v - \bar{u}\mu(x)v - \gamma\bar{u}\partial_{xx}v)dx =$$

$$= [\bar{u}Uv - \gamma\bar{u}\partial_x v]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (-U\partial_x\bar{u}v - \bar{u}\mu(x)v + \gamma\partial_x\bar{u}\partial_x v)dx$$

$$= [\bar{u}Uv - \gamma\bar{u}\partial_x v + \gamma\partial_x\bar{u}v]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (-U\partial_x\bar{u} - \bar{u}\mu(x) - \gamma\partial_{xx}\bar{u})v dx$$

$$= [\bar{u}Uv - \gamma\bar{u}\partial_x v + \gamma\partial_x\bar{u}v]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \overline{(-U\partial_x u - u\overline{\mu(x)} - \gamma\partial_{xx}u)} v dx$$

Hence:

$$\tilde{\mathcal{L}} = -U\partial_x - \overline{\mu(x)} - \gamma\partial_{xx}$$

Adjoint operator
Example with linear PDE and B.C. (1/2)

Space:

Functions $x \in [0,1] \to \mathbb{C}$ such that $u(0) = \partial_x u(1) = 0$.

Scalar-product:

$$< u_1, u_2 > = \int_0^1 u_1^* u_2 dx$$

Linear operator $\mathcal{A}$:

$$\mathcal{A}u = U\partial_x u - \alpha u - \nu \partial_{xx} u$$

Adjoint operator:

$< u_1, \mathcal{A}u_2 > = \int_0^1 u_1^*(U\partial_x u_2 - \alpha u_2 - \nu\partial_{xx} u_2)dx =$

$\int_0^1 (u_1^* U \partial_x u_2 - \alpha u_1^* u_2 - \nu u_1^* \partial_{xx} u_2)dx =$

$[u_1^* U u_2 - \nu u_1^* \partial_x u_2]_0^1 + \int_0^1 (-\partial_x(u_1^* U)u_2 - \alpha u_1^* u_2 + \nu \partial_x u_1^* \partial_x u_2)dx =$

$[u_1^* U u_2 - \nu u_1^* \partial_x u_2 + \nu(\partial_x u_1^*)u_2]_0^1 + \int_0^1 (-\partial_x(U u_1) - \alpha u_1 - \nu \partial_{xx} u_1)^* u_2 dx =$

$< \tilde{\mathcal{A}} u_1, u_2 >$

Hence:

$$\tilde{\mathcal{A}}u = -\partial_x(Uu) - \alpha u - \nu\partial_{xx}u = -U\partial_x u - u\partial_x U - \alpha u - \nu \partial_{xx} u$$

Adjoint operator
Example with linear PDE and B.C. (2/2)

Boundary integral term: $[u_1^* U u_2 - \nu u_1^* \partial_x u_2 + \nu(\partial_x u_1^*)u_2]_0^1 = 0$

At $x = 0$: $u_2 = 0$ and $\partial_x u_2 \neq 0$, so that $u_1 = 0$

At $x = 1$: $\partial_x u_2 = 0$ and $u_2 \neq 0$, so that $u_1^* U + \nu(\partial_x u_1^*) = 0$, or $u_1 U + \nu \partial_x u_1 = 0$

u_1 should be in the following space:

Functions $x \in [0,1] \to \mathbb{C}$ such that $u(0) = u_1(1)U + \nu\partial_x u(1) = 0$.

Outline

- Governing equations
- Asymptotic development
  - ➢ Order ϵ^0 : Base-flow
  - ➢ Order ϵ^1 : Global modes
- Bi-orthogonal basis and adjoint global modes
- Adjoint operator
  - ➢ Definition
  - ➢ Adjoint global modes as solutions of adjoint eigen-problem
- Adjoint linearized Navier-Stokes operator
  - ➢ Adjoint of linearized advection operator
  - ➢ Adjoint of Stokes operator
  - ➢ Adjoint global modes of cylinder flow

Adjoint global modes and biorthogonality

In finite dimension

Theorem:
Let $(\widehat{\mathrm{w}}_\mathrm{i}, \lambda_i)$ be eigenvalues/eigenvectors of $A\widehat{\mathrm{w}}_\mathrm{i} = \lambda_i \widehat{\mathrm{w}}_\mathrm{i}$. Then there exists $(\widetilde{w}_i, \lambda_i^*)$ solution of the adjoint eigenproblem $A^*\widetilde{w}_\mathrm{i} = \lambda_i^* \widetilde{w}_\mathrm{i}$. These structures are the adjoint global modes and may be scaled such that $\widetilde{w}_i^* \widehat{\mathrm{w}}_\mathrm{j} = \delta_{ij}$. The vectors $\widetilde{w}_\mathrm{i}$ are bi-orthogonal with respect to the vectors $\widehat{\mathrm{w}}_\mathrm{j}$.

Adjoint global modes and biorthogonality

In finite dimension

Proof:

$$\lambda_i \hat{w}_i = A\hat{w}_i$$
$$\lambda_j^* \tilde{w}_j = A^* \tilde{w}_j$$
$$\lambda_i \tilde{w}_j^* \hat{w}_i = \tilde{w}_j^* A \hat{w}_i = \left(A^* \tilde{w}_j\right)^* \hat{w}_i = \left(\lambda_j^* \tilde{w}_j\right)^* \hat{w}_i = \lambda_j \tilde{w}_j^* \hat{w}_i$$
$$\left(\lambda_i - \lambda_j\right) \tilde{w}_j^* \hat{w}_i = 0$$

If $\lambda_i \neq \lambda_j$, then $\tilde{w}_j^* \hat{w}_i = 0$

If $\tilde{w}_j^* \hat{w}_i \neq 0$, then $\lambda_i = \lambda_j$.

Conclusion: $\tilde{w}_j$ can be chosen such that

$$\tilde{w}_j^* \hat{w}_i = \delta_{ji}$$

Adjoint global modes and biorthogonality (GL)

Theorem:
Let $(\hat{w}_i, \lambda_i)$ be eigenvalues/eigenvectors of $\lambda_i \hat{w}_i + \mathcal{L}\hat{w}_i = 0$. Then there exists $(\tilde{w}_i, \lambda_i^*)$ solution of the adjoint eigenproblem $\lambda_i^* \tilde{w}_i + \tilde{\mathcal{L}} \tilde{w}_i = 0$. These structures are the adjoint global modes and may be scaled such that $< \tilde{w}_i, \hat{w}_j >= \delta_{ij}$. The vectors $\tilde{w}_i$ are bi-orthogonal with respect to the vectors $\hat{w}_j$.

Adjoint global modes and biorthogonality (GL)

Proof:

$$\lambda_i \widehat{\mathrm{w}}_{\mathrm{i}} + \mathcal{L}\widehat{\mathrm{w}}_{\mathrm{i}} = 0$$
$$\lambda_j^* \widetilde{w}_j + \tilde{\mathcal{L}}\widetilde{w}_j = 0$$
$$< \widetilde{w}_j, \mathcal{L}\widehat{\mathrm{w}}_{\mathrm{i}} > = -\lambda_i < \widetilde{w}_j, \widehat{\mathrm{w}}_{\mathrm{i}} >$$
$$< \widetilde{w}_j, \mathcal{L}\widehat{\mathrm{w}}_{\mathrm{i}} > = < \tilde{\mathcal{L}}\widetilde{w}_j, \widehat{\mathrm{w}}_{\mathrm{i}} > = < -\lambda_j^* \widetilde{w}_j, \widehat{\mathrm{w}}_{\mathrm{i}} > = -\lambda_j < \widetilde{w}_j, \widehat{\mathrm{w}}_{\mathrm{i}} >$$
$$\left(\lambda_i - \lambda_j\right) < \widetilde{w}_j, \widehat{\mathrm{w}}_{\mathrm{i}} > = 0$$

If $\lambda_i \neq \lambda_j$, then $< \widetilde{w}_j, \widehat{\mathrm{w}}_{\mathrm{i}} > = 0$

If $< \widetilde{w}_j, \widehat{\mathrm{w}}_{\mathrm{i}} > \neq 0$, then $\lambda_i = \lambda_j$.

Conclusion: $\widetilde{w}_j$ can be chosen such that

$$< \widetilde{w}_j, \widehat{\mathrm{w}}_{\mathrm{i}} > = \delta_{ji}$$

Adjoint global modes (GL)

Show that: $\widetilde{w}(x) = \xi e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$ with $\xi = \frac{\chi}{\pi^{\frac{1}{2}}\zeta}$ is solution of $\lambda^* \widetilde{w} + \tilde{\mathcal{L}}\widetilde{w} = 0$. Note that the normalization constant ξ has been chosen so that:

$$\langle \widetilde{w}, \widehat{w} \rangle = 1.$$

Can you qualitatively represent $\widehat{w}(x)$ and $\widetilde{w}(x)$?

Noting that:

$$\langle \widetilde{w}, \widetilde{w} \rangle = e^{\frac{U^2}{2\gamma^2 \chi^2}},$$

What is the effect of the advection velocity U on this coefficient?

Nota: $\widetilde{w}_n(x) = \xi_n H_n(\chi x) e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$ are all the adjoint eigenvectors.

Adjoint global modes (GL)

$$\tilde{\mathcal{L}}\widetilde{w} = -\lambda^* \widetilde{w}$$

$$\partial_x \widetilde{w} = (-\frac{U}{2\gamma} - \chi^2 x)\xi e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$$

$$\partial_{xx} \widetilde{w} = \left(-\chi^2 + \frac{U^2}{4\gamma^2} + \chi^4 x^2 + \frac{U}{\gamma}\chi^2 x\right)\xi e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$$

$$\begin{aligned}\tilde{\mathcal{L}}\widetilde{w} &= -\left(U\partial_x \widetilde{w} + \overline{\mu(x)}\widetilde{w} + \gamma \partial_{xx}\widetilde{w}\right) \\ &= -\left(-\frac{U^2}{2\gamma} - U\chi^2 x - i\omega_0 + \mu_0 - \gamma\chi^4 x^2 - \gamma\chi^2 + \frac{U^2}{4\gamma} + \gamma\chi^4 x^2 \right. \\ &\left. + U\chi^2 x\right)\xi e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}} \\ &= -\left(-\frac{U^2}{4\gamma} - i\omega_0 + \mu_0 - \gamma\chi^2\right)\xi e^{-\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}} = -\lambda^* \widetilde{w}\end{aligned}$$

Adjoint global modes and biorthogonality (NS)

Theorem:
Let $(\widehat{\mathrm{w}}_{\mathrm{i}}, \lambda_i)$ be eigenvalues/eigenvectors of $\lambda_i \mathcal{B}\widehat{\mathrm{w}}_{\mathrm{i}} + \left(\mathcal{N}_{w_0} + \mathcal{L}\right)\widehat{\mathrm{w}}_{\mathrm{i}} = 0$. Then there exists $(\widetilde{w}_i, \lambda_i^*)$ solution of the adjoint eigenproblem $\lambda_i^* \mathcal{B}\widetilde{w}_i + \left(\widetilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}}\right)\widetilde{w}_i = 0$. These structures are the adjoint global modes and may be scaled such that $< \widetilde{w}_i, \mathcal{B}\widehat{\mathrm{w}}_{\mathrm{j}} >= \delta_{ij}$. The vectors $\widetilde{w}_{\mathrm{i}}$ are bi-orthogonal with respect to the vectors $\widehat{\mathrm{w}}_{\mathrm{j}}$.

Adjoint global modes and biorthogonality (NS)

Proof:

$$\lambda_i \mathcal{B}\widehat{\mathrm{w}}_{\mathrm{i}} + \left(\mathcal{N}_{w_0} + \mathcal{L}\right)\widehat{\mathrm{w}}_{\mathrm{i}} = 0$$
$$\lambda_j^* \mathcal{B}\widetilde{w}_j + \left(\widetilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}}\right)\widetilde{w}_j = 0$$
$$< \widetilde{w}_j, \left(\mathcal{N}_{w_0} + \mathcal{L}\right)\widehat{\mathrm{w}}_{\mathrm{i}} > = -\lambda_i < \widetilde{w}_j, \mathcal{B}\widehat{\mathrm{w}}_{\mathrm{i}} >$$
$$< \widetilde{w}_j, \left(\mathcal{N}_{w_0} + \mathcal{L}\right)\widehat{\mathrm{w}}_{\mathrm{i}} > =< \left(\widetilde{\mathcal{N}}_{w_0} + \tilde{\mathcal{L}}\right)\widetilde{w}_j, \widehat{\mathrm{w}}_{\mathrm{i}} > =< -\lambda_j^* \mathcal{B}\widetilde{w}_j, \widehat{\mathrm{w}}_{\mathrm{i}} > = -\lambda_j < \widetilde{w}_j, \mathcal{B}\widehat{\mathrm{w}}_{\mathrm{i}} >$$
$$\left(\lambda_i - \lambda_j\right) < \widetilde{w}_j, \mathcal{B}\widehat{\mathrm{w}}_{\mathrm{i}} >= 0$$

If $\lambda_i \neq \lambda_j$, then $< \widetilde{w}_j, \mathcal{B}\widehat{\mathrm{w}}_{\mathrm{i}} > = 0$

If $< \widetilde{w}_j, \mathcal{B}\widehat{\mathrm{w}}_{\mathrm{i}} > \neq 0$, then $\lambda_i = \lambda_j$.

Conclusion: $\widetilde{w}_j$ can be chosen such that

$$< \widetilde{w}_j, \mathcal{B}\widehat{\mathrm{w}}_{\mathrm{i}} > = \delta_{ji}$$

Outline

- Governing equations
- Asymptotic development
  - ➢ Order ϵ^0 : Base-flow
  - ➢ Order ϵ^1 : Global modes
- Bi-orthogonal basis and adjoint global modes
- Adjoint operator
  - ➢ Definition
  - ➢ Adjoint global modes as solutions of adjoint eigen-problem
- Adjoint linearized Navier-Stokes operator
  - ➢ Adjoint of linearized advection operator
  - ➢ Adjoint of Stokes operator
  - ➢ Adjoint global modes of cylinder flow

Adjoint of linearized advection operator (1/4)

Theorem:

Let $\mathcal{N}_{w_0} w = \begin{pmatrix} u \cdot \nabla u_0 + u_0 \cdot \nabla u \\ 0 \end{pmatrix}$ be an operator acting on $w = (u, v, p)$ such that $u = v = 0$ on boundaries. If $< w_1, w_2 > = \iint [u_1^* u_2 + v_1^* v_2 + p_1^* p_2] dxdy$, the adjoint operator of $\mathcal{N}_{w_0}$ is

$$\widetilde{\mathcal{N}}_{w_0} = \begin{bmatrix} \partial_x u_0 & \partial_y u_0 & 0 \\ \partial_x v_0 & \partial_y v_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^* + \begin{bmatrix} -u_0^* \partial_x - v_0^* \partial_y & 0 & 0 \\ 0 & -u_0^* \partial_x - v_0^* \partial_y & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Adjoint of linearized advection operator (2/4)

$$< w_1, \mathcal{N}_{w_0} w_2 > = < \widetilde{\mathcal{N}}_{w_0} w_1, w_2 >$$

$$\iint \begin{bmatrix} u_1^* (u_0 \partial_x u_2 + v_0 \partial_y u_2 + u_2 \partial_x u_0 + v_2 \partial_y u_0) \\ + v_1^* (u_0 \partial_x v_2 + v_0 \partial_y v_2 + u_2 \partial_x v_0 + v_2 \partial_y v_0) \end{bmatrix} dxdy$$

$$= \iint \begin{bmatrix} (u_1^* u_0 \partial_x u_2 + u_1^* v_0 \partial_y u_2 + v_1^* u_0 \partial_x v_2 + v_1^* v_0 \partial_y v_2) \\ + (u_1^* \partial_x u_0 u_2 + u_1^* \partial_y u_0 v_2 + v_1^* \partial_x v_0 u_2 + v_1^* \partial_y v_0 v_2) \end{bmatrix} dxdy$$

$$= \underbrace{\iint [u_1^* u_0 \partial_x u_2 + u_1^* v_0 \partial_y u_2 + v_1^* u_0 \partial_x v_2 + v_1^* v_0 \partial_y v_2] dxdy}_{(*)}$$

$$+ \iint \left[[u_1 \partial_x u_0^* + v_1 \partial_x v_0^*]^* \, u_2 + [u_1 \partial_y u_0^* + v_1 \partial_y v_0^*]^* v_2 \right] dxdy$$

Adjoint of linearized advection operator (3/4)

$$(*) = \overbrace{\int \left[u_1^* u_0 n_x u_2 + u_1^* v_0 n_y u_2 + v_1^* u_0 n_x v_2 + v_1^* v_0 n_y v_2\right] ds}^{0}$$

$$- \iint \left[\partial_x(u_1^* u_0) u_2 + \partial_y(u_1^* v_0) u_2 + \partial_x(v_1^* u_0) v_2 + \partial_y(v_1^* v_0) v_2\right] dxdy$$

$$= - \iint \left[\left[\partial_x(u_1 u_0^*) + \partial_y(u_1 v_0^*)\right]^* u_2 + \left[\partial_x(v_1 u_0^*) + \partial_y(v_1 v_0^*)\right]^* v_2\right] dxdy$$

$$\widetilde{\mathcal{N}}_{w_0} w_1 = \begin{bmatrix} u_1 \partial_x u_0^* + v_1\, \partial_x v_0^* \\ u_1 \partial_y u_0^* + v_1 \partial_y v_0^* \\ 0 \end{bmatrix} + \underbrace{\begin{bmatrix} -\partial_x(u_1 u_0^*) - \partial_y(u_1 v_0^*) \\ -\partial_x(v_1 u_0^*) - \partial_y(v_1 v_0^*) \\ 0 \end{bmatrix}}_{\begin{bmatrix} -u_0^* \partial_x u_1 - v_0^* \partial_y u_1 \\ -u_0^* \partial_x v_1 - v_0^* \partial_y v_1 \\ 0 \end{bmatrix}}$$

(using $\partial_x u_0 + \partial_y v_0 = 0$)

Adjoint of linearized advection operator (4/4)

Conclusion:

$$\widetilde{\mathcal{N}}_{w_0} w_1 = \begin{bmatrix} \partial_x u_0 & \partial_y u_0 & 0 \\ \partial_x v_0 & \partial_y v_0 & 0 \\ 0 & 0 & 0 \end{bmatrix}^* \begin{bmatrix} u_1 \\ v_1 \\ p_1 \end{bmatrix} + \begin{bmatrix} -u_0^* \partial_x - v_0^* \partial_y & 0 & 0 \\ 0 & -u_0^* \partial_x - v_0^* \partial_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ p_1 \end{bmatrix}$$

$$\mathcal{N}_{w_0} w_2 = \begin{bmatrix} \partial_x u_0 & \partial_y u_0 & 0 \\ \partial_x v_0 & \partial_y v_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ p_2 \end{bmatrix} + \begin{bmatrix} u_0 \partial_x + v_0 \partial_y & 0 & 0 \\ 0 & u_0 \partial_x + v_0 \partial_y & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ p_2 \end{bmatrix}$$

$\mathcal{N}_{w_0} \neq \widetilde{\mathcal{N}}_{w_0}$ because of:

- component-type non-normality => $v \rightarrow u$ becomes $u \rightarrow v$
- convective-type non-normality => upstream convection

Outline

- Governing equations
- Asymptotic development
  - Order ϵ^0 : Base-flow
  - Order ϵ^1 : Global modes
- Bi-orthogonal basis and adjoint global modes
- Adjoint operator
  - Definition
  - Adjoint global modes as solutions of adjoint eigen-problem
- Adjoint linearized Navier-Stokes operator
  - Adjoint of linearized advection operator
  - Adjoint of Stokes operator
  - Adjoint global modes of cylinder flow

Adjoint of Stokes operator (1/4)

Theorem:
Let $\mathcal{L} = \begin{pmatrix} -\nu\Delta() & \nabla() \\ -\nabla\cdot() & 0 \end{pmatrix}$ be an operator acting on $\mathrm{w} = (u, v, p)$ such that $u = v = 0$ on boundaries. If $\langle w_1, w_2 \rangle = \iint [u_1^* u_2 + v_1{}^* v_2 + p_1^* p_2] dxdy$, the operator $\mathcal{L}$ is self-afjoint : $\tilde{\mathcal{L}} = \mathcal{L}$.

Adjoint of Stokes operator (2/4)

$$< w_1, \mathcal{L}w_2 >=< \tilde{\mathcal{L}}w_1, w_2 > \qquad \mathcal{L} = \begin{pmatrix} -\nu\Delta() & \nabla() \\ -\nabla\cdot() & 0 \end{pmatrix}$$

$$\iint [u_1^*(-\nu\partial_{xx}u_2 - \nu\partial_{yy}u_2 + \partial_x p_2) + v_1^*(-\nu\partial_{xx}v_2 - \nu\partial_{yy}v_2 + \partial_y p_2)$$
$$+ p_1^*(-\partial_x u_2 - \partial_y v_2)]dxdy$$
$$= \iint [-\nu u_1^*\partial_{xx}u_2 - \nu u_1^*\partial_{yy}u_2 + u_1^*\partial_x p_2 - \nu v_1^*\partial_{xx}v_2 - \nu v_1^*\partial_{yy}v_2$$
$$+ v_1^*\partial_y p_2 - p_1^*\partial_x u_2 - p_1^*\partial_y v_2]dxdy$$
$$= \int [-\nu u_1^* n_x\partial_x u_2 - \nu u_1^* n_y\partial_y u_2 + u_1^* n_x p_2 - \nu v_1^* n_x\partial_x v_2 - \nu v_1^* n_y\partial_y v_2 + v_1^* n_y p_2$$
$$- p_1^* n_x u_2 - p_1^* n_y v_2]ds$$
$$- \iint [-\nu\partial_x u_1^*\partial_x u_2 - \nu\partial_y u_1^*\partial_y u_2 + \partial_x u_1^* p_2 - \nu\partial_x v_1^*\partial_x v_2 - \nu\partial_y v_1^*\partial_y v_2$$
$$+ \partial_y v_1^* p_2 - \partial_x p_1^* u_2 - \partial_y p_1^* v_2]dxdy$$

$u = v = 0$ on boundaries

Adjoint of Stokes operator (3/4)

$$= -\iint [\partial_x u_1^* p_2 + \partial_y v_1^* p_2 - \partial_x p_1^* u_2 - \partial_y p_1^* v_2]dxdy$$
$$+ \underbrace{\iint -[-\nu\partial_x u_1^*\partial_x u_2 - \nu\partial_y u_1^*\partial_y u_2 - \nu\partial_x v_1^*\partial_x v_2 - \nu\partial_y v_1^*\partial_y v_2]dxdy}_{(*)}$$

$$(*) = \int -[-\nu\partial_x u_1^* n_x u_2 - \nu\partial_y u_1^* n_y u_2 - \nu\partial_x v_1^* n_x v_2 - \nu\partial_y v_1^* n_y v_2]ds$$
$$+ \iint [-\nu\partial_{xx}u_1^* u_2 - \nu\partial_{yy}u_1^* u_2 - \nu\partial_{xx}v_1^* v_2 - \nu\partial_{yy}v_1^* v_2]dxdy$$

$$\tilde{\mathcal{L}}w_1 = \begin{bmatrix} -\nu\partial_{xx} - \nu\partial_{yy} & 0 & \partial_x \\ 0 & -\nu\partial_{xx} - \nu\partial_{yy} & \partial_y \\ -\partial_x & -\partial_y & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ p_1 \end{bmatrix}$$

Adjoint of Stokes operator (4/4)

$$\tilde{\mathcal{L}}w_1 = \begin{bmatrix} -\nu\partial_{xx} - \nu\partial_{yy} & 0 & \partial_x \\ 0 & -\nu\partial_{xx} - \nu\partial_{yy} & \partial_y \\ -\partial_x & -\partial_y & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ p_1 \end{bmatrix}$$

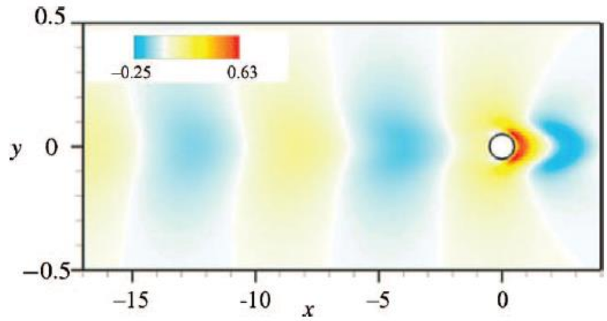
$$\mathcal{L}w_2 = \begin{bmatrix} -\nu\partial_{xx} - \nu\partial_{yy} & 0 & \partial_x \\ 0 & -\nu\partial_{xx} - \nu\partial_{yy} & \partial_y \\ -\partial_x & -\partial_y & 0 \end{bmatrix} \begin{bmatrix} u_2 \\ v_2 \\ p_2 \end{bmatrix}$$

$$\tilde{\mathcal{L}} = \mathcal{L}$$

Outline

- Governing equations
- Asymptotic development
  - Order ϵ^0 : Base-flow
  - Order ϵ^1 : Global modes
- Bi-orthogonal basis and adjoint global modes
- Adjoint operator
  - Definition
  - Adjoint global modes as solutions of adjoint eigen-problem
- Adjoint linearized Navier-Stokes operator
  - Adjoint of linearized advection operator
  - Adjoint of Stokes operator
  - Adjoint global modes of cylinder flow

The adjoint global mode of cylinder flow



Real part of cross-stream velocity field.
Marginal adjoint global mode